

# THETA SERIES ASSOCIATED WITH THE SCHRÖDINGER-WEIL REPRESENTATION

JAE-HYUN YANG

ABSTRACT. In this paper, we define the Schrödinger-Weil representation for the Jacobi group and construct covariant maps for the Schrödinger-Weil representation. Using these covariant maps, we construct Jacobi forms with respect to an arithmetic subgroup of the Jacobi group.

## 1. Introduction

For a given fixed positive integer  $n$ , we let

$$\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \text{Im } \Omega > 0 \}$$

be the Siegel upper half plane of degree  $n$  and let

$$Sp(n, \mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} \mid {}^tgJ_ng = J_n \}$$

be the symplectic group of degree  $n$ , where  $F^{(k,l)}$  denotes the set of all  $k \times l$  matrices with entries in a commutative ring  $F$  for two positive integers  $k$  and  $l$ ,  ${}^tM$  denotes the transposed matrix of a matrix  $M$ ,  $\text{Im } \Omega$  denotes the imaginary part of  $\Omega$  and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that  $Sp(n, \mathbb{R})$  acts on  $\mathbb{H}_n$  transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$  and  $\Omega \in \mathbb{H}_n$ .

For two positive integers  $n$  and  $m$ , we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu {}^t\lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda {}^t\mu' - \mu {}^t\lambda').$$

We let

$$G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \quad (\text{semi-direct product})$$

be the Jacobi group endowed with the following multiplication law

$$\left( g, (\lambda, \mu; \kappa) \right) \cdot \left( g', (\lambda', \mu'; \kappa') \right) = \left( gg', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda} {}^t\mu' - \tilde{\mu} {}^t\lambda') \right)$$

---

Subject Classification: Primary 11F27, 11F50

Keywords and phrases: the Schrödinger-Weil Representation, covariant maps, the Schrödinger representation, the Weil representation, Jacobi forms, Poisson summation formula.

with  $g, g' \in Sp(n, \mathbb{R})$ ,  $(\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n, m)}$  and  $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)g'$ . We let  $\Gamma_n = Sp(n, \mathbb{Z})$  be the Siegel modular group of degree  $n$ . We let

$$\Gamma^J = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n, m)}$$

be the Jacobi modular group. Then we have the *natural action* of  $G^J$  on the Siegel-Jacobi space  $\mathbb{H}_{n, m} := \mathbb{H}_n \times \mathbb{C}^{(m, n)}$  defined by

$$(g, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = (g \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}),$$

where  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n, m)}$  and  $(\Omega, Z) \in \mathbb{H}_{n, m}$ . We refer to [19]-[25] for more details on materials related to the Siegel-Jacobi space.

The Weil representation for the symplectic group was first introduced by A. Weil in [13] to reformulate Siegel's analytic theory of quadratic forms (cf. [12]) in terms of the group theoretical theory. It is well known that the Weil representation plays a central role in the study of the transformation behaviors of the theta series. In this paper, we define the Schrödinger-Weil representation for the Jacobi group  $G^J$ . The aim of this paper is to construct the covariant maps for the Schrödinger-Weil representation, and to construct Jacobi forms with respect to an arithmetic subgroup of  $\Gamma^J$  using these covariant maps.

This paper is organized as follows. In Section 2, we discuss the Schrödinger representation of the Heisenberg group  $H_{\mathbb{R}}^{(n, m)}$  associated with a symmetric nonzero real matrix of degree  $m$ . In Section 3, we review the concept of a Jacobi form briefly. In Section 4, we define the Schrödinger-Weil representation  $\omega_{\mathcal{M}}$  of the Jacobi group  $G^J$  associated with a symmetric positive definite matrix  $\mathcal{M}$  and provide some of the actions of  $\omega_{\mathcal{M}}$  on the representation space  $L^2(\mathbb{R}^{(m, n)})$  explicitly. In Section 5, we construct the covariant maps for the Schrödinger-Weil representation  $\omega_{\mathcal{M}}$ . In the final section we construct Jacobi forms with respect to an arithmetic subgroup of  $\Gamma^J$  using the covariant maps obtained in Section 5.

**Notations:** We denote by  $\mathbb{Z}$  and  $\mathbb{C}$  the ring of integers, and the field of complex numbers respectively.  $\mathbb{C}^\times$  denotes the multiplicative group of nonzero complex numbers.  $T$  denotes the multiplicative group of complex numbers of modulus one. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers  $k$  and  $l$ ,  $F^{(k, l)}$  denotes the set of all  $k \times l$  matrices with entries in a commutative ring  $F$ . For a square matrix  $A \in F^{(k, k)}$  of degree  $k$ ,  $\sigma(A)$  denotes the trace of  $A$ . For any  $M \in F^{(k, l)}$ ,  ${}^tM$  denotes the transposed matrix of  $M$ .  $I_n$  denotes the identity matrix of degree  $n$ . We put  $i = \sqrt{-1}$ . For  $z \in \mathbb{C}$ , we define  $z^{1/2} = \sqrt{z}$  so that  $-\pi/2 < \arg(z^{1/2}) \leq \pi/2$ . Further we put  $z^{\kappa/2} = (z^{1/2})^\kappa$  for every  $\kappa \in \mathbb{Z}$ .

## 2. The Schrödinger Representation of $H_{\mathbb{R}}^{(n, m)}$

First of all, we observe that  $H_{\mathbb{R}}^{(n, m)}$  is a 2-step nilpotent Lie group. The inverse of an element  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n, m)}$  is given by

$$(\lambda, \mu; \kappa)^{-1} = (-\lambda, -\mu; -\kappa + \lambda {}^t\mu - \mu {}^t\lambda).$$

Now we set

$$[\lambda, \mu; \kappa] = (0, \mu; \kappa) \circ (\lambda, 0; 0) = (\lambda, \mu; \kappa - \mu^t \lambda).$$

Then  $H_{\mathbb{R}}^{(n,m)}$  may be regarded as a group equipped with the following multiplication

$$[\lambda, \mu; \kappa] \diamond [\lambda_0, \mu_0; \kappa_0] = [\lambda + \lambda_0, \mu + \mu_0; \kappa + \kappa_0 + \lambda^t \mu_0 + \mu_0^t \lambda].$$

The inverse of  $[\lambda, \mu; \kappa] \in H_{\mathbb{R}}^{(n,m)}$  is given by

$$[\lambda, \mu; \kappa]^{-1} = [-\lambda, -\mu; \kappa + \lambda^t \mu + \mu^t \lambda].$$

We set

$$L = \left\{ [0, \mu; \kappa] \in H_{\mathbb{R}}^{(n,m)} \mid \mu \in \mathbb{R}^{(m,n)}, \kappa = {}^t \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

Then  $L$  is a commutative normal subgroup of  $H_{\mathbb{R}}^{(n,m)}$ . Let  $\widehat{L}$  be the Pontrajagin dual of  $L$ , i.e., the commutative group consisting of all unitary characters of  $L$ . Then  $\widehat{L}$  is isomorphic to the additive group  $\mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R})$  via

$$\langle a, \hat{a} \rangle = e^{2\pi i \sigma(\hat{\mu}^t \mu + \hat{\kappa} \kappa)}, \quad a = [0, \mu; \kappa] \in L, \quad \hat{a} = (\hat{\mu}, \hat{\kappa}) \in \widehat{L},$$

where  $\text{Symm}(m, \mathbb{R})$  denotes the space of all symmetric  $m \times m$  real matrices.

We put

$$S = \left\{ [\lambda, 0; 0] \in H_{\mathbb{R}}^{(n,m)} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}.$$

Then  $S$  acts on  $L$  as follows:

$$\alpha_{\lambda}([0, \mu; \kappa]) = [0, \mu; \kappa + \lambda^t \mu + \mu^t \lambda], \quad [\lambda, 0; 0] \in S.$$

We see that the Heisenberg group  $(H_{\mathbb{R}}^{(n,m)}, \diamond)$  is isomorphic to the semi-direct product  $S \ltimes L$  of  $S$  and  $L$  whose multiplication is given by

$$(\lambda, a) \cdot (\lambda_0, a_0) = (\lambda + \lambda_0, a + \alpha_{\lambda}(a_0)), \quad \lambda, \lambda_0 \in S, \quad a, a_0 \in L.$$

On the other hand,  $S$  acts on  $\widehat{L}$  by

$$\alpha_{\lambda}^*(\hat{a}) = (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda, 0; 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \widehat{L}.$$

Then, we have the relation  $\langle \alpha_{\lambda}(a), \hat{a} \rangle = \langle a, \alpha_{\lambda}^*(\hat{a}) \rangle$  for all  $a \in L$  and  $\hat{a} \in \widehat{L}$ .

We have three types of  $S$ -orbits in  $\widehat{L}$ .

TYPE I. Let  $\hat{\kappa} \in \text{Symm}(m, \mathbb{R})$  be nondegenerate. The  $S$ -orbit of  $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa}) \in \widehat{L}$  is given by

$$\widehat{\mathcal{O}}_{\hat{\kappa}} = \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \widehat{L} \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \cong \mathbb{R}^{(m,n)}.$$

TYPE II. Let  $(\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m,n)} \times \text{Symm}(m, \mathbb{R})$  with degenerate  $\hat{\kappa} \neq 0$ . Then

$$\widehat{\mathcal{O}}_{(\hat{\mu}, \hat{\kappa})} = \left\{ (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}) \mid \lambda \in \mathbb{R}^{(m,n)} \right\} \subsetneq \mathbb{R}^{(m,n)} \times \{\hat{\kappa}\}.$$

TYPE III. Let  $\hat{y} \in \mathbb{R}^{(m,n)}$ . The  $S$ -orbit  $\widehat{\mathcal{O}}_{\hat{y}}$  of  $\hat{a}(\hat{y}) = (\hat{y}, 0)$  is given by

$$\widehat{\mathcal{O}}_{\hat{y}} = \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}).$$

We have

$$\widehat{L} = \left( \bigcup_{\substack{\hat{\kappa} \in \text{Symm}(m, \mathbb{R}) \\ \hat{\kappa} \text{ nondegenerate}}} \widehat{\mathcal{O}}_{\hat{\kappa}} \right) \cup \left( \bigcup_{\hat{y} \in \mathbb{R}^{(m, n)}} \widehat{\mathcal{O}}_{\hat{y}} \right) \cup \left( \bigcup_{\substack{(\hat{\mu}, \hat{\kappa}) \in \mathbb{R}^{(m, n)} \times \text{Symm}(m, \mathbb{R}) \\ \hat{\kappa} \neq 0 \text{ degenerate}}} \widehat{\mathcal{O}}_{(\hat{\mu}, \hat{\kappa})} \right)$$

as a set. The stabilizer  $S_{\hat{\kappa}}$  of  $S$  at  $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$  is given by

$$S_{\hat{\kappa}} = \{0\}.$$

And the stabilizer  $S_{\hat{y}}$  of  $S$  at  $\hat{a}(\hat{y}) = (\hat{y}, 0)$  is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0; 0] \mid \lambda \in \mathbb{R}^{(m, n)} \right\} = S \cong \mathbb{R}^{(m, n)}.$$

In this section, for the present being we set  $H = H_{\mathbb{R}}^{(n, m)}$  for brevity. We see that  $L$  is a closed, commutative normal subgroup of  $H$ . Since  $(\lambda, \mu; \kappa) = (0, \mu; \kappa + \mu^t \lambda) \circ (\lambda, 0; 0)$  for  $(\lambda, \mu; \kappa) \in H$ , the homogeneous space  $X = L \backslash H$  can be identified with  $\mathbb{R}^{(m, n)}$  via

$$Lh = L \circ (\lambda, 0; 0) \longmapsto \lambda, \quad h = (\lambda, \mu; \kappa) \in H.$$

We observe that  $H$  acts on  $X$  by

$$(Lh) \cdot h_0 = L(\lambda + \lambda_0, 0; 0) = \lambda + \lambda_0,$$

where  $h = (\lambda, \mu; \kappa) \in H$  and  $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ .

If  $h = (\lambda, \mu; \kappa) \in H$ , we have

$$l_h = (0, \mu; \kappa + \mu^t \lambda), \quad s_h = (\lambda, 0; 0)$$

in the Mackey decomposition of  $h = l_h \circ s_h$  (cf. [8]). Thus if  $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$ , then we have

$$s_h \circ h_0 = (\lambda, 0; 0) \circ (\lambda_0, \mu_0; \kappa_0) = (\lambda + \lambda_0, \mu_0; \kappa_0 + \lambda^t \mu_0)$$

and so

$$(2.1) \quad l_{s_h \circ h_0} = (0, \mu_0; \kappa_0 + \mu_0^t \lambda_0 + \lambda^t \mu_0 + \mu_0^t \lambda).$$

For a real symmetric matrix  $c = {}^t c \in \text{Symm}(m, \mathbb{R})$  with  $c \neq 0$ , we consider the unitary character  $\chi_c$  of  $L$  defined by

$$(2.2) \quad \chi_c((0, \mu; \kappa)) = e^{\pi i \sigma(c\kappa)} I, \quad (0, \mu; \kappa) \in L,$$

where  $I$  denotes the identity mapping. Then the representation  $\mathscr{W}_c = \text{Ind}_L^H \chi_c$  of  $H$  induced from  $\chi_c$  is realized on the Hilbert space  $H(\chi_c) = L^2(X, d\dot{h}, \mathbb{C}) \cong L^2(\mathbb{R}^{(m, n)}, d\xi)$  as follows. If  $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$  and  $x = Lh \in X$  with  $h = (\lambda, \mu; \kappa) \in H$ , we have

$$(2.3) \quad (\mathscr{W}_c(h_0)f)(x) = \chi_c(l_{s_h \circ h_0})(f(xh_0)), \quad f \in H(\chi_c).$$

It follows from (2.1) that

$$(2.4) \quad (\mathscr{W}_c(h_0)f)(\lambda) = e^{\pi i \sigma\{c(\kappa_0 + \mu_0^t \lambda_0 + 2\lambda^t \mu_0)\}} f(\lambda + \lambda_0),$$

where  $h_0 = (\lambda_0, \mu_0; \kappa_0) \in H$  and  $\lambda \in \mathbb{R}^{(m, n)}$ . Here we identified  $x = Lh$  (resp.  $xh_0 = Lhh_0$ ) with  $\lambda$  (resp.  $\lambda + \lambda_0$ ). The induced representation  $\mathscr{W}_c$  is called the *Schrödinger representation* of  $H$  associated with  $\chi_c$ . Thus  $\mathscr{W}_c$  is a monomial representation.

**Theorem 2.1.** *Let  $c$  be a positive definite symmetric real matrix of degree  $m$ . Then the Schrödinger representation  $\mathcal{W}_c$  of  $H$  is irreducible.*

*Proof.* The proof can be found in [14], Theorem 3.  $\square$

**Remark.** We refer to [14]-[18] for more representations of the Heisenberg group  $H_{\mathbb{R}}^{(n,m)}$  and their related topics.

### 3. Jacobi Forms

Let  $\rho$  be a rational representation of  $GL(n, \mathbb{C})$  on a finite dimensional complex vector space  $V_\rho$ . Let  $\mathcal{M} \in \mathbb{R}^{(m,m)}$  be a symmetric half-integral semi-positive definite matrix of degree  $m$ . Let  $C^\infty(\mathbb{H}_{n,m}, V_\rho)$  be the algebra of all  $C^\infty$  functions on  $\mathbb{H}_{n,m}$  with values in  $V_\rho$ . For  $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ , we define

$$(3.1) \quad \begin{aligned} & (f|_{\rho, \mathcal{M}}[(g, (\lambda, \mu; \kappa))])(\Omega, Z) \\ &= e^{-2\pi i \sigma(\mathcal{M}(Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}C^t(Z + \lambda\Omega + \mu))} \times e^{2\pi i \sigma(\mathcal{M}(\lambda\Omega^t\lambda + 2\lambda^tZ + \kappa + \mu^t\lambda))} \\ & \quad \times \rho(C\Omega + D)^{-1} f(g \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}), \end{aligned}$$

where  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$  and  $(\Omega, Z) \in \mathbb{H}_{n,m}$ .

**Definition 3.1.** Let  $\rho$  and  $\mathcal{M}$  be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} = \{(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu \in \mathbb{Z}^{(m,n)}, \kappa \in \mathbb{Z}^{(m,m)}\}.$$

A *Jacobi form* of index  $\mathcal{M}$  with respect to  $\rho$  on a subgroup  $\Gamma$  of  $\Gamma_n$  of finite index is a holomorphic function  $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$  satisfying the following conditions (A) and (B):

(A)  $f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f$  for all  $\tilde{\gamma} \in \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$ .

(B) For each  $M \in \Gamma_n$ ,  $f|_{\rho, \mathcal{M}}[M]$  has a Fourier expansion of the following form :

$$(f|_{\rho, \mathcal{M}}[M])(\Omega, Z) = \sum_{\substack{T=tT \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(n,m)}} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(T\Omega)} \cdot e^{2\pi i \sigma(RZ)}$$

with a suitable  $\lambda_\Gamma \in \mathbb{Z}$  and  $c(T, R) \neq 0$  only if  $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} tR & \mathcal{M} \end{pmatrix} \geq 0$ .

If  $n \geq 2$ , the condition (B) is superfluous by Koecher principle (cf. [26] Lemma 1.6). We denote by  $J_{\rho, \mathcal{M}}(\Gamma)$  the vector space of all Jacobi forms of index  $\mathcal{M}$  with respect to  $\rho$  on  $\Gamma$ . Ziegler (cf. [26] Theorem 1.8 or [2] Theorem 1.1) proves that the vector space  $J_{\rho, \mathcal{M}}(\Gamma)$  is finite dimensional. In the special case  $\rho(A) = (\det(A))^k$  with  $A \in GL(n, \mathbb{C})$  and a fixed  $k \in \mathbb{Z}$ , we write  $J_{k, \mathcal{M}}(\Gamma)$  instead of  $J_{\rho, \mathcal{M}}(\Gamma)$  and call  $k$  the *weight* of the corresponding Jacobi forms. For more results on Jacobi forms with  $n > 1$  and  $m > 1$ , we refer to [19]-[22] and [26].

**Definition 3.2.** A Jacobi form  $f \in J_{\rho, \mathcal{M}}(\Gamma)$  is said to be a *cuspidal* (or *cuspidal*) form if  $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} tR & \mathcal{M} \end{pmatrix} > 0$  for any  $T, R$  with  $c(T, R) \neq 0$ . A Jacobi form  $f \in J_{\rho, \mathcal{M}}(\Gamma)$  is said to

be *singular* if it admits a Fourier expansion such that a Fourier coefficient  $c(T, R)$  vanishes unless  $\det \begin{pmatrix} \frac{1}{\lambda \Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^t R & \mathcal{M} \end{pmatrix} = 0$ .

We allow a weight  $k$  to be half-integral.

**Definition 3.3.** Let  $\Gamma \subset \Gamma_n$  be a subgroup of finite index. A holomorphic function  $f : \mathbb{H}_{n,m} \rightarrow \mathbb{C}$  is said to be a Jacobi form of a weight  $k \in \frac{1}{2}\mathbb{Z}$  with level  $\Gamma$  and index  $\mathcal{M}$  if it satisfies the following transformation formula

$$(3.2) \quad f(\tilde{\gamma} \cdot (\Omega, Z)) = \chi(\tilde{\gamma}) J_{k,\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) f(\Omega, Z) \quad \text{for all } \tilde{\gamma} \in \tilde{\Gamma} = \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)},$$

where  $\chi$  is a character of  $\tilde{\Gamma}$  and  $J_{k,\mathcal{M}} : \tilde{\Gamma} \times \mathbb{H}_{n,m} \rightarrow \mathbb{C}^\times$  is an automorphic factor defined by

$$J_{k,\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = e^{2\pi i \sigma(\mathcal{M}(Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} C {}^t(Z + \lambda\Omega + \mu))} \\ \times e^{-2\pi i \sigma(\mathcal{M}(\lambda\Omega {}^t\lambda + 2\lambda {}^t Z + \kappa + \mu {}^t\lambda))} \det(C\Omega + D)^k$$

with  $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \tilde{\Gamma}$  with  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{Z}}^{(n,m)}$  and  $(\Omega, Z) \in \mathbb{H}_{n,m}$ .

#### 4. The Schrödinger-Weil Representation

Throughout this section we assume that  $\mathcal{M}$  is a symmetric integral positive definite  $m \times m$  matrix. We consider the Schrödinger representation  $\mathscr{W}_{\mathcal{M}}$  of the Heisenberg group  $H_{\mathbb{R}}^{(n,m)}$  with the central character  $\mathscr{W}_{\mathcal{M}}((0, 0; \kappa)) = \chi_{\mathcal{M}}((0, 0; \kappa)) = e^{\pi i \sigma(\mathcal{M}\kappa)}$ ,  $\kappa \in \text{Symm}(m, \mathbb{R})$  (cf. (2.2)). We note that the symplectic group  $Sp(n, \mathbb{R})$  acts on  $H_{\mathbb{R}}^{(n,m)}$  by conjugation inside  $G^J$ . For a fixed element  $g \in Sp(n, \mathbb{R})$ , the irreducible unitary representation  $\mathscr{W}_{\mathcal{M}}^g$  of  $H_{\mathbb{R}}^{(n,m)}$  defined by

$$(4.1) \quad \mathscr{W}_{\mathcal{M}}^g(h) = \mathscr{W}_{\mathcal{M}}(ghg^{-1}), \quad h \in H_{\mathbb{R}}^{(n,m)}$$

has the property that

$$\mathscr{W}_{\mathcal{M}}^g((0, 0; \kappa)) = \mathscr{W}_{\mathcal{M}}((0, 0; \kappa)) = e^{\pi i \sigma(\mathcal{M}\kappa)} \text{Id}_{H(\chi_{\mathcal{M}})}, \quad \kappa \in \text{Symm}(m, \mathbb{R}).$$

Here  $\text{Id}_{H(\chi_{\mathcal{M}})}$  denotes the identity operator on the Hilbert space  $H(\chi_{\mathcal{M}})$ . According to Stone-von Neumann theorem, there exists a unitary operator  $R_{\mathcal{M}}(g)$  on  $H(\chi_{\mathcal{M}})$  such that  $R_{\mathcal{M}}(g)\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}^g(h)R_{\mathcal{M}}(g)$  for all  $h \in H_{\mathbb{R}}^{(n,m)}$ . We observe that  $R_{\mathcal{M}}(g)$  is determined uniquely up to a scalar of modulus one. From now on, for brevity, we put  $G = Sp(n, \mathbb{R})$ . According to Schur's lemma, we have a map  $c_{\mathcal{M}} : G \times G \rightarrow T$  satisfying the relation

$$R_{\mathcal{M}}(g_1 g_2) = c_{\mathcal{M}}(g_1, g_2) R_{\mathcal{M}}(g_1) R_{\mathcal{M}}(g_2) \quad \text{for all } g_1, g_2 \in G.$$

Therefore  $R_{\mathcal{M}}$  is a projective representation of  $G$  on  $H(\chi_{\mathcal{M}})$  and  $c_{\mathcal{M}}$  defines the cocycle class in  $H^2(G, T)$ . The cocycle  $c_{\mathcal{M}}$  yields the central extension  $G_{\mathcal{M}}$  of  $G$  by  $T$ . The group  $G_{\mathcal{M}}$  is a set  $G \times T$  equipped with the following multiplication

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1 g_2, t_1 t_2 c_{\mathcal{M}}(g_1, g_2)^{-1}), \quad g_1, g_2 \in G, \quad t_1, t_2 \in T.$$

We see immediately that the map  $\tilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow GL(H(\chi_{\mathcal{M}}))$  defined by

$$(4.2) \quad \tilde{R}_{\mathcal{M}}(g, t) = t R_{\mathcal{M}}(g) \quad \text{for all } (g, t) \in G_{\mathcal{M}}$$

is a true representation of  $G_{\mathcal{M}}$ . As in Section 1.7 in [7], we can define the map  $s_{\mathcal{M}} : G \longrightarrow T$  satisfying the relation

$$c_{\mathcal{M}}(g_1, g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1 g_2) \quad \text{for all } g_1, g_2 \in G.$$

Thus we see that

$$G_{2, \mathcal{M}} = \{ (g, t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1} \}$$

is the metaplectic group associated with  $\mathcal{M}$  that is a two-fold covering group of  $G$ . The restriction  $R_{2, \mathcal{M}}$  of  $\tilde{R}_{\mathcal{M}}$  to  $G_{2, \mathcal{M}}$  is the Weil representation of  $G$  associated with  $\mathcal{M}$ . Now we define the projective representation  $\pi_{\mathcal{M}}$  of the Jacobi group  $G^J$  by

$$(4.3) \quad \pi_{\mathcal{M}}(hg) = \mathcal{W}_{\mathcal{M}}(h) R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n, m)}, g \in G.$$

The projective representation  $\pi_{\mathcal{M}}$  of  $G^J$  is naturally extended to the true representation  $\omega_{\mathcal{M}}$  of the group  $G_{2, \mathcal{M}}^J = G_{2, \mathcal{M}} \ltimes H_{\mathbb{R}}^{(n, m)}$ . The representation  $\omega_{\mathcal{M}}$  is called the *Schrödinger-Weil representation* of  $G^J$ . Indeed we have

$$(4.4) \quad \omega_{\mathcal{M}}(h \cdot (g, t)) = t \mathcal{W}_{\mathcal{M}}(h) R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n, m)}, (g, t) \in G_{2, \mathcal{M}}.$$

We recall that the following matrices

$$\begin{aligned} t_0(b) &= \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \text{ with any } b = {}^t b \in \mathbb{R}^{(n, n)}, \\ g_0(\alpha) &= \begin{pmatrix} {}^t \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ with any } \alpha \in GL(n, \mathbb{R}), \\ \sigma_{n, 0} &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \end{aligned}$$

generate the symplectic group  $G = Sp(n, \mathbb{R})$  (cf. [3, p. 326], [10, p. 210]). Therefore the following elements  $h_t(\lambda, \mu; \kappa)$ ,  $t_{\mathcal{M}}(b)$ ,  $g_{\mathcal{M}}(\alpha)$  and  $\sigma_{n, \mathcal{M}}$  of  $G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n, m)}$  defined by

$$\begin{aligned} h_t(\lambda, \mu; \kappa) &= ((I_{2n}, t), (\lambda, \mu; \kappa)) \text{ with } t \in T, \lambda, \mu \in \mathbb{R}^{(m, n)} \text{ and } \kappa \in \mathbb{R}^{(m, m)}, \\ t_{\mathcal{M}}(b) &= ((t_0(b), 1), (0, 0; 0)) \text{ with any } b = {}^t b \in \mathbb{R}^{(n, n)}, \\ g_{\mathcal{M}}(\alpha) &= ((g_0(\alpha), 1), (0, 0; 0)) \text{ with any } \alpha \in GL(n, \mathbb{R}), \\ \sigma_{n, \mathcal{M}} &= ((\sigma_{n, 0}, 1), (0, 0; 0)), \end{aligned}$$

generate the group  $G_{\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n, m)}$ . We can show that the representation  $\tilde{R}_{\mathcal{M}}$  is realized on the representation  $H(\chi_{\mathcal{M}}) = L^2(\mathbb{R}^{(m, n)})$  as follows: for each  $f \in L^2(\mathbb{R}^{(m, n)})$  and  $x \in \mathbb{R}^{(m, n)}$ , the actions of  $\tilde{R}_{\mathcal{M}}$  on the generators are given by

$$(4.5) \quad (\tilde{R}_{\mathcal{M}}(h_t(\lambda, \mu; \kappa))f)(x) = t e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2x {}^t \mu) \}} f(x + \lambda),$$

$$(4.6) \quad \left( \tilde{R}_{\mathcal{M}}(t_{\mathcal{M}}(b))f \right)(x) = e^{\pi i \sigma(\mathcal{M} x b^t x)} f(x),$$

$$(4.7) \quad \left( \tilde{R}_{\mathcal{M}}(g_{\mathcal{M}}(\alpha))f \right)(x) = (\det \alpha)^{\frac{m}{2}} f(x^t \alpha),$$

$$(4.8) \quad \left( \tilde{R}_{\mathcal{M}}(\sigma_{n,\mathcal{M}})f \right)(x) = \left( \frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} f(y) e^{-2\pi i \sigma(\mathcal{M} y^t x)} dy.$$

We denote by  $L_+^2(\mathbb{R}^{(m,n)})$  (resp.  $L_-^2(\mathbb{R}^{(m,n)})$ ) the subspace of  $L^2(\mathbb{R}^{(m,n)})$  consisting of even (resp. odd) functions in  $L^2(\mathbb{R}^{(m,n)})$ . According to Formulas (4.6)-(4.8),  $R_{2,\mathcal{M}}$  is decomposed into representations of  $R_{2,\mathcal{M}}^{\pm}$

$$R_{2,\mathcal{M}} = R_{2,\mathcal{M}}^+ \oplus R_{2,\mathcal{M}}^-,$$

where  $R_{2,\mathcal{M}}^+$  and  $R_{2,\mathcal{M}}^-$  are the even Weil representation and the odd Weil representation of  $G$  that are realized on  $L_+^2(\mathbb{R}^{(m,n)})$  and  $L_-^2(\mathbb{R}^{(m,n)})$  respectively. Obviously the center  $\mathcal{Z}_{2,\mathcal{M}}^J$  of  $G_{2,\mathcal{M}}^J$  is given by

$$\mathcal{Z}_{2,\mathcal{M}}^J = \{((I_{2n}, 1), (0, 0; \kappa)) \in G_{2,\mathcal{M}}^J\} \cong \text{Symm}(m, \mathbb{R}).$$

We note that the restriction of  $\omega_{\mathcal{M}}$  to  $G_{2,\mathcal{M}}$  coincides with  $R_{2,\mathcal{M}}$  and  $\omega_{\mathcal{M}}(h) = \mathcal{W}_{\mathcal{M}}(h)$  for all  $h \in H_{\mathbb{R}}^{(n,m)}$ .

**Remark.** In the case  $n = m = 1$ ,  $\omega_{\mathcal{M}}$  is dealt in [1] and [9]. We refer to [5] and [6] for more details about the Weil representation  $R_{2,\mathcal{M}}$ .

## 5. Covariant Maps for the Schrödinger-Weil representation

As before we let  $\mathcal{M}$  be a symmetric positive definite  $m \times m$  real matrix. We define the mapping  $\mathcal{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2(\mathbb{R}^{(m,n)})$  by

$$(5.1) \quad \mathcal{F}^{(\mathcal{M})}(\Omega, Z)(x) = e^{\pi i \sigma\{\mathcal{M}(x \Omega^t x + 2x^t Z)\}}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}, \quad x \in \mathbb{R}^{(m,n)}.$$

For brevity we put  $\mathcal{F}_{\Omega,Z}^{(\mathcal{M})} := \mathcal{F}^{(\mathcal{M})}(\Omega, Z)$  for  $(\Omega, Z) \in \mathbb{H}_{n,m}$ .

We define the automorphic factor  $J_{\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^{\times}$  for  $G^J$  on  $\mathbb{H}_{n,m}$  by

$$(5.2) \quad J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M}(Z + \lambda \Omega + \mu)(C\Omega + D)^{-1} C^t (Z + \lambda \Omega + \mu))} \\ \times e^{-\pi i \sigma(\mathcal{M}(\lambda \Omega^t \lambda + 2\lambda^t Z + \kappa + \mu^t \lambda))} \det(C\Omega + D)^{\frac{m}{2}},$$

where  $\tilde{g} = (g, (\lambda, \mu; \kappa)) \in G^J$  with  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$  and  $(\Omega, Z) \in \mathbb{H}_{n,m}$ .

**Theorem 5.1.** *The map  $\mathcal{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2(\mathbb{R}^{(m,n)})$  defined by (5.1) is a covariant map for the Schrödinger-Weil representation  $\omega_{\mathcal{M}}$  of  $G^J$  and the automorphic factor  $J_{\mathcal{M}}$  for  $G^J$  on  $\mathbb{H}_{n,m}$  defined by Formula (5.2). In other words,  $\mathcal{F}^{(\mathcal{M})}$  satisfies the following covariance relation*

$$(5.3) \quad \omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} = J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g} \cdot (\Omega, Z)}^{(\mathcal{M})}$$

for all  $\tilde{g} \in G^J$  and  $(\Omega, Z) \in \mathbb{H}_{n, m}$ .

*Proof.* For an element  $\tilde{g} = (g, (\lambda, \mu; \kappa)) \in G^J$  with  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ , we put  $(\Omega_*, Z_*) = \tilde{g} \cdot (\Omega, Z)$  for  $(\Omega, Z) \in \mathbb{H}_{n, m}$ . Then we have

$$\begin{aligned} \Omega_* &= g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}. \end{aligned}$$

In this section we use the notations  $t_0(b)$ ,  $g_0(\alpha)$  and  $\sigma_{n, 0}$  in Section 4. Since the following elements  $h(\lambda, \mu; \kappa)$ ,  $t(b)$ ,  $g(\alpha)$  and  $\sigma_n$  of  $G^J$  defined by

$$\begin{aligned} h(\lambda, \mu; \kappa) &= (I_{2n}, (\lambda, \mu; \kappa)) \quad \text{with } \lambda, \mu \in \mathbb{R}^{(m, n)}, \kappa \in \mathbb{R}^{(m, m)}, \\ t(b) &= (t_0(b), (0, 0; 0)) \quad \text{with } b = {}^t b \in \mathbb{R}^{(m, m)}, \\ g(\alpha) &= (g_0(\alpha), (0, 0; 0)) \quad \text{with } \alpha \in GL(n, \mathbb{R}), \\ \sigma_n &= (\sigma_{n, 0}, (0, 0; 0)) \end{aligned}$$

generate the Jacobi group, it suffices to prove the covariance relation (5.3) for the above generators.

**Case I.**  $\tilde{g} = h(\lambda, \mu; \kappa)$  with  $\lambda, \mu \in \mathbb{R}^{(m, n)}$ ,  $\kappa \in \mathbb{R}^{(m, m)}$ .

In this case, we have

$$\Omega_* = \Omega, \quad Z_* = Z + \lambda\Omega + \mu$$

and

$$J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = e^{-\pi i \sigma \{ \mathcal{M}(\lambda \Omega {}^t \lambda + 2 \lambda {}^t Z + \kappa + \mu {}^t \lambda) \}}.$$

According to Formula (4.5), for  $x \in \mathbb{R}^{(m, n)}$ ,

$$\begin{aligned} & \left( \omega_{\mathcal{M}}(h(\lambda, \mu; \kappa)) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\ &= e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2 x {}^t \mu) \}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(x + \lambda) \\ &= e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2 x {}^t \mu) \}} e^{\pi i \sigma \{ \mathcal{M}((x + \lambda) \Omega {}^t (x + \lambda) + 2 (x + \lambda) {}^t Z) \}}. \end{aligned}$$

On the other hand, according to Formula (5.2), for  $x \in \mathbb{R}^{(m, n)}$ ,

$$\begin{aligned} & J_{\mathcal{M}}(h(\lambda, \mu; \kappa), (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g} \cdot (\Omega, Z)}^{(\mathcal{M})}(x) \\ &= J_{\mathcal{M}}(h(\lambda, \mu; \kappa), (\Omega, Z))^{-1} \mathcal{F}_{\Omega, Z + \lambda\Omega + \mu}^{(\mathcal{M})}(x) \\ &= e^{\pi i \sigma \{ \mathcal{M}(\lambda \Omega {}^t \lambda + 2 \lambda {}^t Z + \kappa + \mu {}^t \lambda) \}} \cdot e^{\pi i \sigma \{ \mathcal{M}(x \Omega {}^t x + 2 x {}^t (Z + \lambda\Omega + \mu)) \}} \\ &= e^{\pi i \sigma \{ \mathcal{M}(\kappa + \mu {}^t \lambda + 2 x {}^t \mu) \}} e^{\pi i \sigma \{ \mathcal{M}((x + \lambda) \Omega {}^t (x + \lambda) + 2 (x + \lambda) {}^t Z) \}}. \end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case  $\tilde{g} = h(\lambda, \mu; \kappa)$  with  $\lambda, \mu, \kappa$  real.

**Case II.**  $\tilde{g} = t(b)$  with  $b = {}^t b \in \mathbb{R}^{(n, n)}$ .

In this case, we have

$$\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = 1.$$

According to Formula (4.6), we obtain

$$\left( \omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) = e^{\pi i \sigma(\mathcal{M} x b^t x)} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} (x), \quad x \in \mathbb{R}^{(m, n)}.$$

On the other hand, according to Formula (5.2), for  $x \in \mathbb{R}^{(m, n)}$ , we obtain

$$\begin{aligned} & J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g}, (\Omega, Z)}^{(\mathcal{M})} (x) \\ &= \mathcal{F}_{\Omega+b, Z}^{(\mathcal{M})} (x) \\ &= e^{\pi i \sigma(\mathcal{M}(x(\Omega+b)^t x + 2x^t Z))} \\ &= e^{\pi i \sigma(\mathcal{M} x b^t x)} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} (x). \end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case  $\tilde{g} = t(b)$  with  $b = {}^t b \in \mathbb{R}^{(n, n)}$ .

**Case III.**  $\tilde{g} = g(\alpha)$  with  $\alpha \in GL(n, \mathbb{R})$ .

In this case, we have

$$\Omega_* = {}^t \alpha \Omega \alpha, \quad Z_* = Z \alpha$$

and

$$J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = (\det \alpha)^{-\frac{m}{2}}.$$

According to Formula (4.7), for  $x \in \mathbb{R}^{(m, n)}$ ,

$$\begin{aligned} & \left( \omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\ &= (\det \alpha)^{\frac{m}{2}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} (x {}^t \alpha) \\ &= (\det \alpha)^{\frac{m}{2}} \cdot e^{\pi i \sigma\{\mathcal{M}(x {}^t \alpha \Omega {}^t (x {}^t \alpha) + 2x {}^t \alpha {}^t Z)\}}. \end{aligned}$$

On the other hand, according to Formula (5.2), for  $x \in \mathbb{R}^{(m, n)}$ ,

$$\begin{aligned} & J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g}, (\Omega, Z)}^{(\mathcal{M})} (x) \\ &= (\det \alpha)^{\frac{m}{2}} \mathcal{F}_{{}^t \alpha \Omega \alpha, Z \alpha}^{(\mathcal{M})} (x) \\ &= (\det \alpha)^{\frac{m}{2}} \cdot e^{\pi i \sigma\{\mathcal{M}(x {}^t \alpha \Omega {}^t (x {}^t \alpha) + 2x {}^t \alpha {}^t Z)\}}. \end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case  $\tilde{g} = g(\alpha)$  with  $\alpha \in GL(n, \mathbb{R})$ .

**Case IV.**  $\tilde{g} = \left( \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, (0, 0; 0) \right)$ .

In this case, we have

$$\Omega_* = -\Omega^{-1}, \quad Z_* = Z \Omega^{-1}$$

and

$$J_{\mathcal{M}}(\tilde{g}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)} (\det \Omega)^{\frac{m}{2}}.$$

In order to prove the covariance relation (5.3), we need the following useful lemma.

**Lemma 5.1.** *For a fixed element  $\Omega \in \mathbb{H}_n$  and a fixed element  $Z \in \mathbb{C}^{(m,n)}$ , we obtain the following property*

$$(5.4) \quad \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} dx_{11} \cdots dx_{mn} = \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(Z \Omega^{-1} Z)},$$

where  $x = (x_{ij}) \in \mathbb{R}^{(m,n)}$ .

*Proof of Lemma 5.1.* By a simple computation, we see that

$$e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} = e^{-\pi i \sigma(Z \Omega^{-1} Z)} \cdot e^{\pi i \sigma\{(x + Z \Omega^{-1}) \Omega^t (x + Z \Omega^{-1})\}}.$$

Since the real Jacobi group  $Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(m,n)}$  acts on  $\mathbb{H}_{n,m}$  holomorphically, we may put

$$\Omega = i A^t A, \quad Z = i V, \quad A \in \mathbb{R}^{(n,n)}, \quad V = (v_{ij}) \in \mathbb{R}^{(m,n)}.$$

Then we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(x \Omega^t x + 2x^t Z)} dx_{11} \cdots dx_{mn} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{(x + iV(iA^t A)^{-1}) (iA^t A)^t \{x + iV(iA^t A)^{-1}\}\}} dx_{11} \cdots dx_{mn} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{(x + V(A^t A)^{-1}) A^t A^t \{x + V(A^t A)^{-1}\}\}} dx_{11} \cdots dx_{mn} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma\{(uA)^t (uA)\}} du_{11} \cdots du_{mn} \quad (\text{Put } u = x + V(A^t A)^{-1} = (u_{ij})) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} \int_{\mathbb{R}^{(m,n)}} e^{-\pi \sigma(w^t w)} (\det A)^{-m} dw_{11} \cdots dw_{mn} \quad (\text{Put } w = uA = (w_{ij})) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} (\det A)^{-m} \cdot \left( \prod_{i=1}^m \prod_{j=1}^n \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} \right) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} (\det A)^{-m} \quad (\text{because } \int_{\mathbb{R}} e^{-\pi w_{ij}^2} dw_{ij} = 1 \text{ for all } i, j) \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} (\det (A^t A))^{-\frac{m}{2}} \\ &= e^{-\pi i \sigma(Z \Omega^{-1} Z)} \left( \det \left( \frac{\Omega}{i} \right) \right)^{-\frac{m}{2}}. \end{aligned}$$

This completes the proof of Lemma 5.1. □

According to Formula (4.8), for  $x \in \mathbb{R}^{(m,n)}$ , we obtain

$$\begin{aligned} & \left( \omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\ &= \left( \frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(y) e^{-2\pi i \sigma(\mathcal{M} y^t x)} dy \\ &= \left( \frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{\mathcal{M}(y \Omega^t y + 2y^t Z)\}} e^{-2\pi i \sigma(\mathcal{M} y^t x)} dy \\ &= \left( \frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma\{\mathcal{M}(y \Omega^t y + 2y^t (Z - x))\}} dy. \end{aligned}$$

If we substitute  $u = \mathcal{M}^{\frac{1}{2}} y$ , then  $du = (\det \mathcal{M})^{\frac{n}{2}} dy$ . Therefore according to Lemma 5.1, we obtain

$$\begin{aligned}
& \left( \omega_{\mathcal{M}}(\tilde{g}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \right) (x) \\
&= \left( \frac{1}{i} \right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m, n)}} e^{\pi i \sigma(u \Omega^t u + 2 \mathcal{M}^{1/2} u^t (Z-x))} (\det \mathcal{M})^{-\frac{n}{2}} du \\
&= \left( \frac{1}{i} \right)^{\frac{mn}{2}} \int_{\mathbb{R}^{(m, n)}} e^{\pi i \sigma(u \Omega^t u + 2 u^t (\mathcal{M}^{1/2} (Z-x)))} du \\
&= \left( \frac{1}{i} \right)^{\frac{mn}{2}} \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma\{\mathcal{M}^{1/2} (Z-x) \Omega^{-1} {}^t (Z-x) \mathcal{M}^{1/2}\}} \quad (\text{by Lemma 5.1}) \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} (Z-x) \Omega^{-1} {}^t (Z-x))} \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} (Z \Omega^{-1} {}^t Z + x \Omega^{-1} {}^t x - 2 Z \Omega^{-1} {}^t x))}.
\end{aligned}$$

On the other hand, according to Formula (5.2), for  $x \in \mathbb{R}^{(m, n)}$ ,

$$\begin{aligned}
& J_{\mathcal{M}}(\tilde{g}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{g}, (\Omega, Z)}^{(\mathcal{M})}(x) \\
&= e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)} (\det \Omega)^{-\frac{m}{2}} \mathcal{F}_{-\Omega^{-1}, Z \Omega^{-1}}^{(\mathcal{M})}(x) \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)} e^{\pi i \sigma\{\mathcal{M} (x (-\Omega^{-1})^t x + 2 x^t (Z \Omega^{-1}))\}} \\
&= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} (Z \Omega^{-1} {}^t Z + x \Omega^{-1} {}^t x - 2 Z \Omega^{-1} {}^t x))}.
\end{aligned}$$

Therefore we prove the covariance relation (5.3) in the case  $\tilde{g} = \sigma_n$ . Since  $J_{\mathcal{M}}$  is an automorphic factor for  $G^J$  on  $\mathbb{H}_{n, m}$ , we see that if the covariance relation (5.3) holds for two elements  $\tilde{g}_1, \tilde{g}_2$  in  $G^J$ , then it holds for  $\tilde{g}_1 \tilde{g}_2$ . Finally we complete the proof.  $\square$

## 6. Construction of Jacobi Forms

Let  $(\pi, V_{\pi})$  be a unitary representation of  $G^J$  on the representation space  $V_{\pi}$ . We assume that  $(\pi, V_{\pi})$  satisfies the following conditions (A) and (B):

(A) There exists a vector valued map

$$\mathcal{F} : \mathbb{H}_{n, m} \longrightarrow V_{\pi}, \quad (\Omega, Z) \mapsto \mathcal{F}_{\Omega, Z} := \mathcal{F}(\Omega, Z)$$

satisfying the following covariance relation

$$(6.1) \quad \pi(\tilde{\gamma}) \mathcal{F}_{\Omega, Z} = \psi(\tilde{\gamma}) J(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)} \quad \text{for all } \tilde{\gamma} \in G^J, (\Omega, Z) \in \mathbb{H}_{n, m},$$

where  $\psi$  is a character of  $G^J$  and  $J : G^J \times \mathbb{H}_{n, m} \longrightarrow GL(1, \mathbb{C})$  is a certain automorphic factor for  $G^J$  on  $\mathbb{H}_{n, m}$ .

(B) Let  $\tilde{\Gamma}$  be an arithmetic subgroup of  $\Gamma^J$ . There exists a linear functional  $\theta : V_{\pi} \longrightarrow \mathbb{C}$  which is semi-invariant under the action of  $\tilde{\Gamma}$ , in other words, for all  $\tilde{\gamma} \in \tilde{\Gamma}$  and  $(\Omega, Z) \in \mathbb{H}_{n, m}$ ,

$$(6.2) \quad \langle \pi^*(\tilde{\gamma}) \theta, \mathcal{F}_{\Omega, Z} \rangle = \langle \theta, \pi(\tilde{\gamma})^{-1} \mathcal{F}_{\Omega, Z} \rangle = \chi(\tilde{\gamma}) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle,$$

where  $\pi^*$  is the contragredient of  $\pi$  and  $\chi : \tilde{\Gamma} \longrightarrow T$  is a unitary character of  $\tilde{\Gamma}$ .

Under the assumptions (A) and (B) on a unitary representation  $(\pi, V_\pi)$ , we define the function  $\Theta$  on  $\mathbb{H}_{n,m}$  by

$$(6.3) \quad \Theta(\Omega, Z) := \langle \theta, \mathcal{F}_{\Omega, Z} \rangle = \theta(\mathcal{F}_{\Omega, Z}), \quad (\Omega, Z) \in \mathbb{H}_{n,m}.$$

We now shall see that  $\Theta$  is an automorphic form on  $\mathbb{H}_{n,m}$  with respect to  $\tilde{\Gamma}$  for the automorphic factor  $J$ .

**Lemma 6.1.** *Let  $(\pi, V_\pi)$  be a unitary representation of  $G^J$  satisfying the above assumptions (A) and (B). Then the function  $\Theta$  on  $\mathbb{H}_{n,m}$  defined by (6.3) satisfies the following modular transformation behavior*

$$(6.4) \quad \Theta(\tilde{\gamma} \cdot (\Omega, Z)) = \psi(\tilde{\gamma})^{-1} \chi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \Theta(\Omega, Z)$$

for all  $\tilde{\gamma} \in \tilde{\Gamma}$  and  $(\Omega, Z) \in \mathbb{H}_{n,m}$ .

*Proof.* For any  $\tilde{\gamma} \in \tilde{\Gamma}$  and  $(\Omega, Z) \in \mathbb{H}_{n,m}$ , according to the assumptions (6.1) and (6.2), we obtain

$$\begin{aligned} \Theta(\tilde{\gamma} \cdot (\Omega, Z)) &= \langle \theta, \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)} \rangle \\ &= \langle \theta, \psi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \pi(\tilde{\gamma}) \mathcal{F}_{\Omega, Z} \rangle \\ &= \psi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \langle \theta, \pi(\tilde{\gamma}) \mathcal{F}_{\Omega, Z} \rangle \\ &= \psi(\tilde{\gamma})^{-1} \chi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \langle \theta, \mathcal{F}_{\Omega, Z} \rangle \\ &= \psi(\tilde{\gamma})^{-1} \chi(\tilde{\gamma})^{-1} J(\tilde{\gamma}, (\Omega, Z)) \Theta(\Omega, Z). \end{aligned}$$

□

Now for a positive definite integral symmetric matrix  $\mathcal{M}$  of degree  $m$ , we define the holomorphic function  $\Theta_{\mathcal{M}} : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$  by

$$(6.5) \quad \Theta_{\mathcal{M}}(\Omega, Z) := \sum_{\xi \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}(\xi \Omega^t \xi + 2\xi^t Z))}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}.$$

**Theorem 6.1.** *Let  $\mathcal{M}$  be a symmetric positive definite, unimodular even integral matrix of degree  $m$ . Then for any  $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$  with  $\gamma \in \Gamma_n$  and  $(\lambda, \mu; \kappa) \in H_{\mathbb{Z}}^{(n,m)}$ , the function  $\Theta_{\mathcal{M}}$  satisfies the functional equation*

$$(6.6) \quad \Theta_{\mathcal{M}}(\tilde{\gamma} \cdot (\Omega, Z)) = \rho_{\mathcal{M}}(\tilde{\gamma}) J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) \Theta_{\mathcal{M}}(\Omega, Z), \quad (\Omega, Z) \in \mathbb{H}_{n,m},$$

where  $\rho_{\mathcal{M}}(\tilde{\gamma})$  is a uniquely determined character of  $\Gamma^J$  with  $|\rho_{\mathcal{M}}(\tilde{\gamma})|^8 = 1$  and  $J_{\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$  is the automorphic factor for  $G^J$  on  $\mathbb{H}_{n,m}$  defined by the formula (5.2).

*Proof.* For an element  $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$  with  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$  and  $(\lambda, \mu; \kappa) \in H_{\mathbb{Z}}^{(n,m)}$ , we put  $(\Omega_*, Z_*) = \tilde{\gamma} \cdot (\Omega, Z)$  for  $(\Omega, Z) \in \mathbb{H}_{n,m}$ . Then we have

$$\begin{aligned} \Omega_* &= \gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}. \end{aligned}$$

We define the linear functional  $\vartheta$  on  $L^2(\mathbb{R}^{(m,n)})$  by

$$\vartheta(f) = \langle \vartheta, f \rangle := \sum_{\xi \in \mathbb{Z}^{(m,n)}} f(\xi), \quad f \in L^2(\mathbb{R}^{(m,n)}).$$

We note that  $\Theta_{\mathcal{M}}(\Omega, Z) = \vartheta(\mathcal{F}_{\Omega, Z}^{(\mathcal{M})})$ . Since  $\mathcal{F}^{(\mathcal{M})}$  is a covariant map for the Schrödinger-Weil representation  $\omega_{\mathcal{M}}$  by Theorem 5.1, according to Lemma 6.1, it suffices to prove that  $\vartheta$  is semi-invariant for  $\omega_{\mathcal{M}}$  under the action of  $\Gamma^J$ , in other words,  $\vartheta$  satisfies the following semi-invariance relation

$$(6.7) \quad \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle = \rho_{\mathcal{M}}(\tilde{\gamma})^{-1} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle$$

for all  $\tilde{\gamma} \in \Gamma^J$  and  $(\Omega, Z) \in \mathbb{H}_{n,m}$ .

We see that the following elements  $h(\lambda, \mu; \kappa)$ ,  $t(b)$ ,  $g(\alpha)$  and  $\sigma_n$  of  $\Gamma^J$  defined by

$$\begin{aligned} h(\lambda, \mu; \kappa) &= (I_{2n}, (\lambda, \mu; \kappa)) \text{ with } \lambda, \mu \in \mathbb{Z}^{(m,n)} \text{ and } \kappa \in \mathbb{Z}^{(m,m)}, \\ t(b) &= (t_0(b), (0, 0; 0)) \text{ with any } b = {}^t b \in \mathbb{Z}^{(n,n)}, \\ g(\alpha) &= (g_0(\alpha), (0, 0; 0)) \text{ with any } \alpha \in GL(n, \mathbb{Z}), \\ \sigma_n &= (s_{n,0}, (0, 0; 0)) \end{aligned}$$

generate the Jacobi modular group  $\Gamma^J$ . Therefore it suffices to prove the semi-invariance relation (6.7) for the above generators of  $\Gamma^J$ .

**Case I.**  $\tilde{\gamma} = h(\lambda, \mu; \kappa)$  with  $\lambda, \mu \in \mathbb{Z}^{(m,n)}$ ,  $\kappa \in \mathbb{Z}^{(m,m)}$ .

In this case, we have

$$\Omega_* = \Omega, \quad Z_* = Z + \lambda \Omega + \mu$$

and

$$J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = e^{-\pi i \sigma\{\mathcal{M}(\lambda \Omega^t \lambda + 2 \lambda^t Z + \kappa + \mu^t \lambda)\}}.$$

According to the covariance relation (5.3),

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma}(\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \langle \vartheta, \mathcal{F}_{\Omega, Z + \lambda \Omega + \mu}^{(\mathcal{M})} \rangle \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma\{\mathcal{M}(A \Omega^t A + 2 A^t (Z + \lambda \Omega + \mu))\}} \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \cdot e^{-\pi i \sigma(\mathcal{M}(\lambda \Omega^t \lambda + 2 \lambda^t Z))} \\ & \quad \times \sum_{A \in \mathbb{Z}^{(m,n)}} e^{2 \pi i \sigma(\mathcal{M} A^t \mu)} e^{\pi i \sigma\{\mathcal{M}((A + \lambda) \Omega^t (A + \lambda) + 2 (A + \lambda)^t Z)\}} \\ &= e^{\pi i \sigma(\mathcal{M}(\kappa + \mu^t \lambda))} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we used the fact that  $\sigma(\mathcal{M} A^t \mu)$  is an integer. We put  $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(h(\lambda, \mu; \kappa)) = e^{-\pi i \sigma(\mathcal{M}(\kappa + \mu^t \lambda))}$ . Therefore  $\vartheta$  satisfies the semi-invariance relation (6.7) in the case  $\tilde{\gamma} = h(\lambda, \mu; \kappa)$  with  $\lambda, \mu \in \mathbb{Z}^{(m,n)}$ ,  $\kappa \in \mathbb{Z}^{(m,m)}$ .

**Case II.**  $\tilde{\gamma} = t(b)$  with  $b = {}^t b \in \mathbb{Z}^{(n,n)}$ .

In this case, we have

$$\Omega_* = \Omega + b, \quad Z_* = Z \quad \text{and} \quad J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = 1.$$

According to the covariance relation (5.3), we obtain

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma}, (\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, \mathcal{F}_{\Omega+b, Z}^{(\mathcal{M})} \rangle \\ &= \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A(\Omega+b) {}^t A + 2 A {}^t Z) \}} \\ &= \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A \Omega {}^t A + 2 A {}^t Z) \}} \cdot e^{\pi i \sigma \{ \mathcal{M} A b {}^t A \}} \\ &= \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A \Omega {}^t A + 2 A {}^t Z) \}} \\ &= \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we used the fact that  $\sigma(\mathcal{M} A b {}^t A)$  is an even integer. We put  $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(t(b)) = 1$ . Therefore  $\vartheta$  satisfies the semi-invariance relation (6.7) in the case  $\tilde{\gamma} = t(b)$  with  $b = {}^t b \in \mathbb{Z}^{(n,n)}$ .

**Case III.**  $\tilde{\gamma} = g(\alpha)$  with  $\alpha \in GL(n, \mathbb{Z})$ .

In this case, we have

$$\Omega_* = {}^t \alpha \Omega \alpha, \quad Z_* = Z \alpha$$

and

$$J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = (\det \alpha)^{-\frac{m}{2}}.$$

According to the covariance relation (5.3), we obtain

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma}, (\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= (\det \alpha)^{\frac{m}{2}} \langle \vartheta, \mathcal{F}_{{}^t \alpha \Omega \alpha, Z \alpha}^{(\mathcal{M})} \rangle \\ &= (\det \alpha)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}_{{}^t \alpha \Omega \alpha, Z \alpha}^{(\mathcal{M})}(A) \\ &= (\det \alpha)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma \{ \mathcal{M}(A {}^t \alpha \Omega {}^t (A {}^t \alpha) + 2 A {}^t \alpha {}^t Z) \}} \\ &= (\det \alpha)^{\frac{m}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we put  $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(g(\alpha)) = (\det \alpha)^{-\frac{m}{2}}$ . Therefore  $\vartheta$  satisfies the semi-invariance relation (6.7) in the case  $\tilde{\gamma} = g(\alpha)$  with  $\alpha \in GL(n, \mathbb{Z})$ .

**Case IV.**  $\tilde{\gamma} = \left( \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, (0, 0; 0) \right)$ .

In this case, we have

$$\Omega_* = -\Omega^{-1}, \quad Z_* = Z \Omega^{-1}$$

and

$$J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z)) = e^{\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)} (\det \Omega)^{\frac{m}{2}}.$$

In the process of the proof of Theorem 5.1, using Lemma 5.1, we already showed that

$$(6.8) \quad \int_{\mathbb{R}^{(m,n)}} e^{\pi i \sigma(\mathcal{M}(y \Omega {}^t y + 2 y {}^t Z))} dy = (\det \mathcal{M})^{-\frac{n}{2}} \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)}.$$

By (6.8), we see that the Fourier transform of  $\mathcal{F}_{\Omega, Z}^{(\mathcal{M})}$  is given by

$$(6.9) \quad \widehat{\mathcal{F}_{\Omega, Z}^{(\mathcal{M})}}(x) = (\det \mathcal{M})^{-\frac{n}{2}} \left( \det \frac{\Omega}{i} \right)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M}(Z-x) \Omega^{-1} {}^t (Z-x))}.$$

According to the covariance relation (5.3), Formula (6.9) and Poisson summation formula, we obtain

$$\begin{aligned} & \langle \vartheta, \omega_{\mathcal{M}}(\tilde{\gamma}) \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= \langle \vartheta, J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \mathcal{F}_{\tilde{\gamma} \cdot (\Omega, Z)}^{(\mathcal{M})} \rangle \\ &= J_{\mathcal{M}}(\tilde{\gamma}, (\Omega, Z))^{-1} \langle \vartheta, \mathcal{F}_{-\Omega^{-1}, Z \Omega^{-1}}^{(\mathcal{M})} \rangle \\ &= (\det \Omega)^{-\frac{m}{2}} e^{-\pi i \sigma(\mathcal{M} Z \Omega^{-1} {}^t Z)} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(\mathcal{M}(A \Omega^{-1} {}^t A - 2 A \Omega^{-1} {}^t Z))} \\ &= (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(\mathcal{M}(Z \Omega^{-1} {}^t Z + A \Omega^{-1} {}^t A - 2 A \Omega^{-1} {}^t Z))} \\ &= (\det \Omega)^{-\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} e^{-\pi i \sigma(\mathcal{M}(Z-A) \Omega^{-1} {}^t (Z-A))} \\ &= (\det \Omega)^{-\frac{m}{2}} (\det \mathcal{M})^{\frac{n}{2}} \left( \det \frac{\Omega}{i} \right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \widehat{\mathcal{F}_{\Omega, Z}^{(\mathcal{M})}}(A) \quad (\text{by Formula (6.9)}) \\ &= (\det \mathcal{M})^{\frac{n}{2}} \left( \det \frac{I_n}{i} \right)^{\frac{m}{2}} \sum_{A \in \mathbb{Z}^{(m,n)}} \mathcal{F}_{\Omega, Z}^{(\mathcal{M})}(A) \quad (\text{by Poisson summation formula}) \\ &= (\det \mathcal{M})^{\frac{n}{2}} (-i)^{\frac{mn}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle \\ &= (-i)^{\frac{mn}{2}} \langle \vartheta, \mathcal{F}_{\Omega, Z}^{(\mathcal{M})} \rangle. \end{aligned}$$

Here we used the fact that  $\det \mathcal{M} = 1$  because  $\mathcal{M}$  is unimodular. We put  $\rho_{\mathcal{M}}(\tilde{\gamma}) = \rho_{\mathcal{M}}(\sigma_n) = (-i)^{-\frac{mn}{2}}$ . Therefore  $\vartheta$  satisfies the semi-invariance relation (6.7) in the case  $\tilde{\gamma} = \sigma_n$ . The proof of Case IV is completed. Since  $J_{\mathcal{M}}$  is an automorphic factor for  $G^J$  on  $\mathbb{H}_{n,m}$ , we see that if the formula (6.6) holds for two elements  $\tilde{\gamma}_1, \tilde{\gamma}_2$  in  $\Gamma^J$ , then it holds for  $\tilde{\gamma}_1 \tilde{\gamma}_2$ . Finally we complete the proof of Theorem 6.1.  $\square$

**Remark 6.1.** For a symmetric positive definite integral matrix  $\mathcal{M}$  that is not unimodular even integral, we obtain a similar transformation formula like (6.6). If  $m$  is odd,  $\Theta_{\mathcal{M}}(\Omega, Z)$  is a Jacobi form of a half-integral weight  $\frac{m}{2}$  and index  $\frac{\mathcal{M}}{2}$  with respect to a suitable arithmetic subgroup  $\Gamma_{\Theta, \mathcal{M}}^J$  of  $\Gamma^J$  and a character  $\rho_{\mathcal{M}}$  of  $\Gamma_{\Theta, \mathcal{M}}^J$ .

For instance, we obtain the following :

**Theorem 6.2.** Let  $\mathcal{M}$  be a symmetric positive definite integral matrix of degree  $m$  such that  $\det(\mathcal{M}) = 1$ . Let  $\Gamma_{1,2}$  be an arithmetic subgroup of  $\Gamma_n$  generated by all the following elements

$$t(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix}, \quad g(\alpha) = \begin{pmatrix} {}^t\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad \sigma_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where  $b = {}^tb \in \mathbb{Z}^{(n,n)}$  with even diagonal and  $\alpha \in \mathbb{Z}^{(n,n)}$ . We put

$$\Gamma_{1,2}^J := \Gamma_{1,2} \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

Then  $\Theta_{\mathcal{M}}$  satisfies the transformation formula (6.6) for all  $\tilde{\gamma} \in \Gamma_{1,2}^J$ . Therefore  $\Theta_{\mathcal{M}}$  is a Jacobi form of weight  $\frac{m}{2}$  with level  $\Gamma_{1,2}$  and index  $\frac{\mathcal{M}}{2}$  for the uniquely determined character  $\rho_{\mathcal{M}}$  of  $\Gamma_{1,2}^J$ .

*Proof.* The proof is essentially the same as the proof of Theorem 6.1. We leave the detail to the reader.  $\square$

## REFERENCES

- [1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser, 1998.
- [2] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math., **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
- [3] E. Freitag, *Siegelsche Modulformen*, Grundlehren der mathematischen Wissenschaften **55**, Springer-Verlag, Berlin-Heidelberg-New York (1983).
- [4] E. Hecke, *Herleitung des Euler-Produktes der Zetafunktion und einiger L-Reihen aus ihrer Funktionalgleichung*, Math. Ann. **119** (1944), 266-287 (=Werke, 919-940).
- [5] S. Gelbart, *Weil's Representation and the Spectrum of the Metaplectic Group*, Lecture Notes in Math. **530**, Springer-Verlag, Berlin and New York, 1976.
- [6] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil Representations and Harmonic Polynomials*, Invent. Math. **44** (1978), 1-47.
- [7] G. Lion and M. Vergne, *The Weil representation, Maslov index and Theta series*, Progress in Math., **6**, Birkhäuser, Boston, Basel and Stuttgart, 1980.
- [8] G. W. Mackey, *Induced Representations of Locally Compact Groups I*, Ann. of Math., **55** (1952), 101-139.
- [9] J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. of Math., **158** (2003), 419-471.
- [10] D. Mumford, *Tata Lectures on Theta I*, Progress in Math. **28**, Boston-Basel-Stuttgart (1983).
- [11] G. Shimura, *On modular forms of half integral weight*, Ann. of Math., **97** (1973), 440-481; Collected Papers, 1967-1977, Vol. II, Springer-Verlag (2002), 532-573.
- [12] C. L. Siegel, *Indefinite quadratische Formen und Funktionentheorie I and II*, Math. Ann. **124** (1951), 17-54 and Math. Ann. **124** (1952), 364-387; Gesammelte Abhandlungen, Band III, Springer-Verlag (1966), 105-142 and 154-177.

- [13] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math., **111** (1964), 143–211; Collected Papers (1964–1978), Vol. III, Springer-Verlag (1979), 1–69.
- [14] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., **123** (1991), 103–117.
- [15] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II*, J. Number Theory, **49** (1) (1994), 63–72.
- [16] J.-H. Yang, *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, the Mathematical Society of Japan, New Series, **22** (1) (1996), 37–49.
- [17] J.-H. Yang, *Fock Representations of the Heisenberg Group  $H_{\mathbb{R}}^{(g,h)}$* , J. Korean Math. Soc., **34**, no. 2 (1997), 345–370.
- [18] J.-H. Yang, *Lattice Representations of the Heisenberg Group  $H_{\mathbb{R}}^{(g,h)}$* , Math. Annalen, **317** (2000), 309–323.
- [19] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.
- [20] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33–58.
- [21] J.-H. Yang, *Singular Jacobi forms*, Trans. of American Math. Soc. **347**, No. 6 (1995), 2041–2049.
- [22] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. **47** (6) (1995), 1329–1339.
- [23] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space*, Houston Journal of Mathematics, Vol. **32**, No. 3 (2006), 701–712.
- [24] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space*, Journal of Number Theory, **127** (2007), 83–102 or arXiv:math.NT/0507215.
- [25] J.-H. Yang, *A partial Cayley transform of Siegel-Jacobi disk*, J. Korean Math. Soc. **45**, No. 3 (2008), 781–794.
- [26] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg **59** (1989), 191–224.

DEPARTMENT OF MATHEMATICS, INHA UNIVERSITY, INCHEON 402-751, KOREA  
*E-mail address:* `jhyang@inha.ac.kr`